Properties of Walrasian Demand

Econ 3030

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Lecture 5

Outline

1. Properties of Walrasian Demand
2. Indirect Utility Function
3. Envelope Theorem

Summary of Constrained Optimization

- When $\mathbf{x}^*$ solves $\boxed{\max f(\mathbf{x}) \quad \text{subject to} \quad \begin{array}{r} g_i(\mathbf{x}) \leq 0 \\ i = 1, .., m \end{array}}$ then

$$\nabla L(\mathbf{x}^*, \boldsymbol{\lambda}) = \nabla f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}$$

and $g_i(\mathbf{x}^*) \leq 0$, $\lambda_i \geq 0$, and $\lambda_i g_i(\mathbf{x}^*) = 0$ for $i = 1, ..., m$.

Summary of Constrained Optimization

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  and $g_i(\mathbf{x}^*) \leq 0$, $\lambda_i \geq 0$, and $\lambda_i g_i(\mathbf{x}^*) = 0$ for $i = 1, ..., m$.

- Important details:
  - If the better than set or the constraint sets are not convex: big trouble.
  - If functions are not differentiable: small trouble.
    - If the geometry still works we can find a more general theorem (see convex analysis).
- When does this fail?
  - If the constraint qualification condition fails.
  - If the objective function is not quasi concave.
- This means you must check the second order conditions when in doubt.

Walrasian Demand

Definition

Given a utility function $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, the Walrasian demand correspondence $x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n_+$ is defined by

$$x^*(\mathbf{p}, w) = \arg \max_{\mathbf{x} \in B_{p,w}} u(\mathbf{x}) \qquad \text{where} \qquad B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

- When the utility function is quasi-concave and differentiable, the First Order Conditions for utility maximization say:

$$\nabla\left(\begin{array}{c}\text{utility}\\\text{function}\end{array}\right) - \lambda_{\substack{\text{budget}\\\text{constraint}}} \nabla\left(\begin{array}{c}\text{budget}\\\text{constraint}\end{array}\right) - \sum \lambda_{\substack{\text{non negativity}\\\text{constraints}}} \nabla\left(\begin{array}{c}\text{non negativity}\\\text{constraints}\end{array}\right) = \mathbf{0}$$

- So, at a solution $\mathbf{x}^* \in x^*(\mathbf{p}, w)$:

$$\frac{\partial L(\mathbf{x}^*)}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda_w x_i^* + \lambda_i = 0 \text{ for all } i = 1, .., n$$

Marginal Rate of Substitution

- Suppose we have an optimal consumption bundle $\mathbf{x}^*$ where some goods are consumed in strictly positive amounts.
- Then, the corresponding non-negativity constraints hold and the correspoding multipliers equal 0.
- At such a solution $\mathbf{x}^*$, the first order condition is:

$$\frac{\partial u\left(\mathbf{x}^*\right)}{\partial x_i} = \lambda_w p_i \qquad \text{for all } i \text{ such that } x_i^* > 0$$

- This expression implies

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}}{\frac{\partial u(\mathbf{x}^*)}{\partial x_k}} = \frac{p_j}{p_k} \qquad \text{for any } j, k \text{ such that } x_j^*, x_k^* > 0$$

- Marginal rates of substitution equal price ratios at an optimum.
- Geometrically: the slope of indifference curve equals the slope of the budget set.

Upper Contour Set and Budget Set

- At an optimal consumption bundle $\mathbf{x}^*$ where goods are consumed in strictly positive amounts:

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}}{\frac{\partial u(\mathbf{x}^*)}{\partial x_k}} = \frac{p_j}{p_k} \qquad \text{for any } j, k \text{ such that } x_j^*, x_k^* > 0$$

Marginal Utility Per Dollar Spent

- At an optimal consumption bundle $\mathbf{x}^*$ where good i is consumed in strictly positive amount, the first order condition is:

$$\frac{\partial u\left(\mathbf{x}^*\right)}{\partial x_i} = \lambda_w p_i \qquad \text{for all } i \text{ such that } x_i^* > 0$$

- Rearranging, one obtains

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}}{p_j} = \frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_k}}{p_k} \qquad \text{for any } j, k \text{ such that } x_j^*, x_k^* > 0$$

  the marginal utility per dollar spent must be equal across goods.

  - If not, there are j and k for which $\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}}{p_j} < \frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_k}}{p_k}$
  - DM can buy $\frac{\varepsilon}{p_j}$ less of j, and $\frac{\varepsilon}{p_k}$ more of k, so the budget constraint still holds, and
  - by Taylor's theorem, the utility at the new choice is

$$u\left(\mathbf{x}^*\right) + \frac{\partial u\left(\mathbf{x}^*\right)}{\partial x_j}\left(-\frac{\varepsilon}{p_j}\right) + \frac{\partial u\left(\mathbf{x}^*\right)}{\partial x_k}\left(\frac{\varepsilon}{p_k}\right) + o\left(\varepsilon\right) = u\left(\mathbf{x}^*\right) + \varepsilon\left(\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_k}}{p_k} - \frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}}{p_j}\right) + o\left(\varepsilon\right)$$

  which implies that $\mathbf{x}^*$ is not an optimum.

- Think about the case in which some goods are consumed in zero amount.

Demand and Indirect Utility Function

The Walrasian demand correspondence $x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ is defined by

$$x^*(\mathbf{p}, w) = \arg \max_{\mathbf{x} \in B(\mathbf{p},w)} u(\mathbf{x}) \qquad \text{where} \qquad B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

Definition

Given a continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the indirect utility function $v : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$v(\mathbf{p}, w) = u(\mathbf{x}^*) \qquad \text{where} \qquad \mathbf{x}^* \in x^*(\mathbf{p}, w).$$

- The indirect utility function gives the optimized value of the objective function as prices and wages change and the consumer adjusts her optimal choice accordingly.

Results

- The Walrasian demand correspondence is upper hemi continuous
  - To prove this we need properties that characterize continuity for correspondences.
- The indirect utility function is continuous.
  - To prove this we need properties that characterize continuity for correspondences.

Berge's Theorem of the Maximum

- The theorem of the maximum lets us establish the previous two results.

Theorem (Theorem of the Maximum)

If $f : X \to \mathbb{R}$ is a continuous function and $\varphi : Q \to X$ is a continuous correspondence with nonempty and compact values, then

- *the mapping $x^* : Q \to X$ defined by*

$$x^*(\mathbf{q}) = \arg \max_{x \in \varphi(\mathbf{q})} f(\mathbf{x})$$

  is an upper hemicontinuous correspondence and

- *the mapping $v : Q \to \mathbb{R}$ defined by*

$$v(\mathbf{q}) = \max_{x \in \varphi(\mathbf{q})} f(\mathbf{x})$$

  is a continuous function.

- Berge's Theorem is useful when exogenous parameters enter the optimization problem only through the constraints, and do not directly enter the objective function.
  - In the consumer's problem prices and income do not enter the utility function, they only affect the budget set.

Continuity for Correspondences

Reminder from math camp.

Definition

A correspondence $\varphi : X \to Y$ is

- upper hemicontinuous at $\mathbf{x} \in X$ if for any neighborhood $V \subseteq Y$ containing $\varphi(\mathbf{x})$, there exists a neighborhood $U \subseteq X$ of $\mathbf{x}$ such that $\varphi(\mathbf{x}') \subseteq V$ for all $\mathbf{x}' \in U$.
- lower hemicontinuous at $\mathbf{x} \in X$ if for any neighborhood $V \subseteq Y$ such that $\varphi(\mathbf{x}) \cap V \neq \emptyset$, there exists a neighborhood $U \subseteq X$ of $\mathbf{x}$ such that $\varphi(\mathbf{x}') \cap V \neq \emptyset$ for all $x' \in U$.
- A correspondence is upper (lower) hemicontinuous if it is upper (lower) hemicontinuous for all $\mathbf{x} \in X$.
- A correspondence is continuous if it is both upper and lower hemicontinuous.

Continuity for Correspondences: Examples

Exercise

Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by:

$$\varphi(x) = \begin{cases} \{1\} & \text{if } x < 1 \\ [0,2] & \text{if } x \geq 1 \end{cases}.$$

Prove that φ is upper hemicontinuous, but not lower hemicontinuous.

Exercise

Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by:

$$\varphi(x) = \begin{cases} \{1\} & \text{if } x \leq 1 \\ [0,2] & \text{if } x > 1 \end{cases}.$$

Prove that φ is lower hemicontinuous, but not upper hemicontinuous.

Properties of Walrasian Demand

Proposition

If $u(\mathbf{x})$ is continuous, then $x^*(\mathbf{p}, w)$ is upper hemicontinuous and $v(\mathbf{p}, w)$ is continuous.

Proof.

Apply Berge's Theorem:

If $u : \mathbb{R}^n_+ \to \mathbb{R}$ a continuous function and $B(\mathbf{p}, w) : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ is a continuous correspondence with nonempty and compact values. Then:

(i): $x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ defined by $x^*(\mathbf{p}, w) = \arg\max_{\mathbf{x} \in B(\mathbf{p}, w)} u(\mathbf{x})$ is an upper hemicontinuous correspondence and

(ii): $v : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ defined by $v(\mathbf{p}, w) = \max_{\mathbf{x} \in B(\mathbf{p}, w)} u(\mathbf{x})$ is a continuous function.

- We need continuity of the correspondence from price-wage pairs to budget sets.
  - We must show that $B : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ defined by

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}$$

is continuous and we are done. □

Continuity of the Budget Set Correspondence

Exercise

Show that the correspondence from price-wage pairs to budget sets, $B : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ defined by

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}$$

is continuous.

- First show that $B(\mathbf{p}, w)$ is upper hemi continuous.
- Then show that $B(\mathbf{p}, w)$ is lower hemi continuous.
- There was a very very similar problem in math camp.

Properties of Walrasian Demand

Definitions

Given a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the Walrasian demand correspondence $x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ and the indirect utility function $v : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ are defined by

$$x^*(\mathbf{p}, w) = \arg\max_{\mathbf{x} \in B_{\mathbf{p},w}} u(\mathbf{x}) \qquad \text{where} \qquad B_{\mathbf{p},w} = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

$$v(\mathbf{p}, w) = u(\mathbf{x}^*) \qquad \text{where} \qquad \mathbf{x}^* \in x^*(\mathbf{p}, w)$$

Properties of Walrasian demand and indirect utility function:

- if u is continuous, then $x^*(\mathbf{p}, w)$ is nonempty and compact
- $x^*(\mathbf{p}, w)$ is homogeneous of degree zero: for any $\alpha > 0$, $x^*(\alpha\mathbf{p}, \alpha w) = x^*(\mathbf{p}, w)$
- if u represents a locally nonsatiated $\succsim$, then $\mathbf{p} \cdot \mathbf{x} = w$ for any $\mathbf{x} \in x^*(\mathbf{p}, w)$
- if u is quasiconcave, then $x^*(\mathbf{p}, w)$ is convex
- if u is strictly quasiconcave, then $x^*(\mathbf{p}, w)$ is unique
- if $u(\mathbf{x})$ is continuous, then $x^*(\mathbf{p}, w)$ is upper hemicontinuous
- if $u(\mathbf{x})$ is continuous, then $v(\mathbf{p}, w)$ is continuous.

Comparative Statics

We want to answer the following fundamental question: *How does Walrasian Demand change as income and/or (some) prices change?*

- To do this we need familiar tools: the implicit function theorem and the envelope theorem
- The implicit function theorem tells how a function's optimizer responds to changes in the parameters
  - In other words, how an endogenous variable changes when the exogenous variables change
- The envelope theorem tells how the optimized value of a function respond to changes in the parameters
  - In other words, how the optimal value changes when the exogenous variables change
- In math camp, you have seen the version of this without constraints; now, we adapt those results so that they apply to constrained optimization problems.

Implicit Function Theorem (same as math camp)

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$.

Theorem (Implicit Function Theorem)

Suppose A is an open set in $\mathbb{R}^{n+m}$ and $f : A \to \mathbb{R}^n$ is continuously differentiable. Let $D_{\mathbf{x}} f$ be the derivative of f with respect to $\mathbf{x}$ (an $n \times n$ matrix).
If $f(\overline{\mathbf{x}}, \overline{\mathbf{q}}) = \mathbf{0}_n$ and $D_{\mathbf{x}} f(\overline{\mathbf{x}}, \overline{\mathbf{q}})$ is nonsingular, then there exists a neighborhood B of $\overline{\mathbf{q}}$ in $\mathbb{R}^m$ and a unique continuously differentiable $g : B \to \mathbb{R}^n$ such that

$$g(\overline{\mathbf{q}}) = \overline{\mathbf{x}} \qquad \text{and} \qquad f(g(\mathbf{q}), \mathbf{q}) = \mathbf{0}_n \text{ for all } \mathbf{q} \in B$$

Moreover,

$$D_{\mathbf{q}} g(\overline{\mathbf{q}}) = -\left[D_{\mathbf{x}} f(\overline{\mathbf{x}}, \overline{\mathbf{q}})\right]^{-1} D_{\mathbf{q}} f(\overline{\mathbf{x}}, \overline{\mathbf{q}}).$$

- Notation: $(D_{\mathbf{x}} f)_{ij} = \frac{\partial f_i}{\partial x_j}$
- The theorem gives a way to write, locally, $\mathbf{x}$ as dependent on $\mathbf{q}$ via a differentiable implicit function.
- We apply this to optimization problems by setting $f(\cdot)$ to be the corresponding FOC;
- In consumer theory $g(\cdot)$ is Walrasian demand (a function of prices and income).

Bordered Hessian

- For comparative statics the following object is useful.

Definition

Let $m = \{i : x_i^*(\overline{\mathbf{p}}, \bar{w}) > 0\}$ and reindex $\mathbb{R}^n$ so these m dimensions come first. The bordered Hessian H of u with respect to its first m dimensions is

$$H = \begin{bmatrix} 0 & (D_{\mathbf{x}}u)^\top \\ D_{\mathbf{x}}u & D^2_{\mathbf{xx}}u \end{bmatrix} = \begin{bmatrix} 0 & u_1 & u_2 & \cdots & u_m \\ u_1 & u_{11} & u_{21} & \cdots & u_{m1} \\ u_2 & u_{12} & u_{22} & \cdots & u_{m2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m & u_{1m} & u_{2m} & \cdots & u_{mm} \end{bmatrix}, \quad \text{where } u_i = \frac{\partial u}{\partial x_i} \text{ and } u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

- This only considers demand functions which are not zero and then takes first and second derivatives of the utility function with respect to those goods.

Differentiability of Walrasian Demand

Proposition

Suppose u is twice continuously differentiable, locally nonsatiated, strictly quasiconcave, and that there exists $\varepsilon > 0$ such that $x_i^*(\overline{\mathbf{p}}, \bar{w}) > 0$ if and only if $x_i^*(\mathbf{p}, w) > 0$, for all $(\mathbf{p}, w)$ such that $\|(\overline{\mathbf{p}}, \bar{w}) - (\mathbf{p}, w)\| < \varepsilon$.[a] If H is nonsingular at $(\overline{\mathbf{p}}, \bar{w})$, then $x^*(\mathbf{p}, w)$ is continuously differentiable at $(\overline{\mathbf{p}}, \bar{w})$.

[a] This condition is automatically satisfied if $x_i^* > 0$ for all i, by continuity of x^*.

- These are sufficient conditions for differentiability of $x^*(\mathbf{p}, w)$, yet we do not have axioms on $\succsim$ that deliver a continuously differentiable utility.
- We also assume demand is strictly positive (locally), but this is the very object we want to characterize.
- This is "bad" math.

Proof.

We want to use the Implicit Funtion Theorem. Things we know right away:

- u is strictly quasiconcave $\Rightarrow x^*(\mathbf{p}, w)$ is a function.
- u is continuously differentiable $\Rightarrow$ the first order conditions are well defined.
- u is locally nonsatiated $\Rightarrow$ the budget constraint binds.
- $x^*(\mathbf{p}, w)$ is strictly positive in the first m commodities $\Rightarrow$ ignore corresponding nonnegativity constraints.
- It now suffices to show that $x^*(\mathbf{p}, w)$ is differentiable in the first m prices, and w, and ignore the last $n - m$ commodities.[a]
- The remainder of the proof uses IFT to show that $x^*(\mathbf{p}, w)$ is differentiable as desired;
- Fill in the details. □

[a] There is a neighborhood of $(\overline{\mathbf{p}}, \bar{w})$ where consumption is zero for the last $n - m$ commodities, and these constant dimensions will have no effect on the differentiability of x^*.

Comparative Statics

Without constraints: this is the same as math camp

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$.

Problem

Let $\phi(\mathbf{x};\mathbf{q})$ be a function $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and let $x^*(\mathbf{q})$ be the solution to $\max \phi(\mathbf{x};\mathbf{q})$.

Choose $\mathbf{x}$ to maximize ϕ, and write the solution as a function of the parameters $\mathbf{q}$.

Comparative Statics

We want to know how the optimal choice $x^*(\mathbf{q})$ changes as the parameters $\mathbf{q}$ change: what is the derivative of $x^*(\mathbf{q})$?

- If ϕ is strictly concave and differentiable the Implicit Function Theorem gives the answer.
  - The implicit function we use in the theorem is defined by the solution to the first order conditions of the optimization problem.

Envelope Theorem Without Constraints

Define: $x^*(\mathbf{q}) = \arg\max \phi(\mathbf{x};\mathbf{q})$. *The maximizer solves the FOC:* $f(\mathbf{x};\mathbf{q}) \equiv D_{\mathbf{x}}\phi(\mathbf{x};\mathbf{q}) = \mathbf{0}_n^\top$.
The Jacobian of f *is* $D_{\mathbf{x}} f(\mathbf{x};\mathbf{q}) = D_{\mathbf{xx}}\phi(\mathbf{x};\mathbf{q})$ *, and is nonsingular (why?).*

Fix $\overline{\mathbf{q}}$ and the corresponding $x^*(\overline{\mathbf{q}})$, and note that this solutions solves the FOC: $f(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}}) = \mathbf{0}_n^\top$

Reminder from math camp: by IFT close to $\overline{\mathbf{q}}$

$$D_{\mathbf{q}} x^*(\overline{\mathbf{q}}) = \underbrace{-\left[D_{\mathbf{x}} f(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}})\right]^{-1} D_{\mathbf{q}} f(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}})}_{\text{by IFT}} = -\underbrace{\left[D_{\mathbf{xx}}\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}})\right]^{-1}}_{n\times n} \underbrace{D_{\mathbf{qx}}\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}})}_{n\times m}$$

- Using the Chain Rule, the change in the optimized value is:

$$D_{\mathbf{q}}\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}}) = D_{\mathbf{q}}\phi(\mathbf{x},\mathbf{q})\Big|_{\substack{\mathbf{q}=\overline{\mathbf{q}}\\ \mathbf{x}=x^*(\overline{\mathbf{q}})}} + \overbrace{D_{\mathbf{x}}\phi(\mathbf{x},\mathbf{q})\Big|_{\substack{\mathbf{q}=\overline{\mathbf{q}}\\ \mathbf{x}=x^*(\overline{\mathbf{q}})}}}^{=0} D_{\mathbf{q}} x^*(\mathbf{q}) = D_{\mathbf{q}}\phi(\mathbf{x},\mathbf{q})\Big|_{\substack{\mathbf{q}=\overline{\mathbf{q}}\\ \mathbf{x}=x^*(\overline{\mathbf{q}})}}$$

  - the "second order effect" (how the maximizer responds to $\mathbf{q}$) is irrelevant; only the "first order effect" matters (how the objective evaluated at the fixed maximizer changes with $\mathbf{q}$); this is the Envelope Theorem.

- If $\mathbf{x}$ and $\mathbf{q}$ are scalars, $D_{\mathbf{q}} x^*(\mathbf{q})$ becomes $\frac{\partial x^*}{\partial q} = -\frac{\frac{\partial^2\phi}{\partial q\partial x}}{\frac{\partial^2\phi}{\partial x^2}}$.

Comparative Statics With Constraints

There are k equality constraints, $F_i(\mathbf{x}; \mathbf{q}) = 0$ where each F_i is a smooth function.

Problem

The maximization now becomes: $$x^*(\mathbf{q}) = \arg \max_{F(\mathbf{x};\mathbf{q})=\mathbf{0}_k} \phi(\mathbf{x}; \mathbf{q})$$

Assume constraint qualifications are met and form the Lagrangian:

$$L(\boldsymbol{\lambda}, \mathbf{x}; \mathbf{q}) = \phi(\mathbf{x}; \mathbf{q}) - \boldsymbol{\lambda}^\top F(\mathbf{x}; \mathbf{q}).$$

- The derivative of the Lagrangian w.r.t. the choice variables is:

$$\underbrace{f(\boldsymbol{\lambda}, \mathbf{x}; \mathbf{q})}_{1\times(k+n)} \equiv D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda}, \mathbf{x}; \mathbf{q}) = \begin{pmatrix} \overbrace{-F(\mathbf{x}; \mathbf{q})}^{1\times k} \\ \\ \underbrace{D_\mathbf{x}\phi(\mathbf{x}; \mathbf{q})}_{1\times n} - \underbrace{\boldsymbol{\lambda}^\top}_{1\times k} \underbrace{D_\mathbf{x} F(\mathbf{x}; \mathbf{q})}_{k\times n} \end{pmatrix}$$

- Fix $\overline{\mathbf{q}} \in \mathbb{R}^m$ and let $\boldsymbol{\lambda}^*$ and $\mathbf{x}^*$ be the corresponding maximum;
- As before, we use IFT on the FOC $f(\boldsymbol{\lambda}^*, \mathbf{x}^*; \mathbf{q}) = \mathbf{0}_{k+n}^\top$ (the Jacobian is non singular).

Comparative Statics With Constraints: IFC

By IFT, $x^*(\mathbf{q})$ and $\lambda^*(\mathbf{q})$ are implicit functions of $\mathbf{q}$ in a neighborhood of $\bar{\mathbf{q}}$, and

$$D_{\mathbf{q}}(\lambda^*(\bar{\mathbf{q}}), x^*(\bar{\mathbf{q}})) = -\left[D_{(\boldsymbol{\lambda},\mathbf{x})} f(\boldsymbol{\lambda}^*, \mathbf{x}^*; \bar{\mathbf{q}})\right]^{-1} D_{\mathbf{q}} f(\boldsymbol{\lambda}^*, \mathbf{x}^*; \bar{\mathbf{q}})$$

- Since f is the derivative of the Lagrangian: $\underbrace{f(\boldsymbol{\lambda}^*, \mathbf{x}^*; \bar{\mathbf{q}})}_{1\times(k+n)} = \begin{pmatrix} -F(\mathbf{x};\mathbf{q}) \\ D_{\mathbf{x}}\phi(\mathbf{x};\mathbf{q}) - \boldsymbol{\lambda}^\top D_{\mathbf{x}} F(\mathbf{x};\mathbf{q}) \end{pmatrix}$

  Thus

$$\underbrace{D_{(\boldsymbol{\lambda},\mathbf{x})} f(\boldsymbol{\lambda}^*, \mathbf{x}^*; \bar{\mathbf{q}})}_{(k+n)\times(k+n)} = \begin{pmatrix} \overbrace{\mathbf{0}}^{k\times k} & \overbrace{-D_{\mathbf{x}} F(\mathbf{x}^*;\bar{\mathbf{q}})}^{k\times n} \\ \underbrace{-D_{\mathbf{x}} F(\mathbf{x}^*;\bar{\mathbf{q}})^\top}_{n\times k} & \underbrace{D^2_{\mathbf{xx}}\phi(\mathbf{x}^*;\bar{\mathbf{q}}) - D_{\mathbf{x}}[\boldsymbol{\lambda}^\top D_{\mathbf{x}} F(\mathbf{x}^*;\bar{\mathbf{q}})]}_{n\times n} \end{pmatrix}$$

  and

$$\underbrace{D_{\mathbf{q}} f(\boldsymbol{\lambda}^*, \mathbf{x}^*; \bar{\mathbf{q}})}_{(k+n)\times m} = \begin{pmatrix} \overbrace{-D_{\mathbf{q}} F(x^*(\bar{\mathbf{q}});\bar{\mathbf{q}})}^{k\times m} \\ \underbrace{D^2_{\mathbf{qx}}\phi(\mathbf{x}^*;\bar{\mathbf{q}})}_{n\times m} - \underbrace{D_{\mathbf{q}}(\boldsymbol{\lambda}^\top D_{\mathbf{x}} F(\mathbf{x}^*;\bar{\mathbf{q}})^\top)}_{n\times m} \end{pmatrix}$$

Envelope Theorem With Constraints

Now we can figure out the change in the objective function.

- Using the Chain Rule, the change of $\phi^*(\mathbf{q}) = \phi(x^*(\mathbf{q}); \mathbf{q})$ is:

$$D_{\mathbf{q}}\phi(x^*(\overline{\mathbf{q}}); \overline{\mathbf{q}}) = \left. D_{\mathbf{q}}\phi(\mathbf{x}, \mathbf{q}) \right|_{\mathbf{q}=\mathbf{q},\ \mathbf{x}=x^*(\overline{\mathbf{q}})} + \left. D_{\mathbf{x}}\phi(\mathbf{x}, \mathbf{q}) \right|_{\mathbf{q}=\overline{\mathbf{q}},\ \mathbf{x}=x^*(\overline{\mathbf{q}})} D_{\mathbf{q}}x^*(\mathbf{q})$$

- Because the constraint holds, $-F(\mathbf{x}; \mathbf{q}) = \mathbf{0}_k$ and thus

$$-D_{\mathbf{x}}F(\mathbf{x}; \mathbf{q}) D_{\mathbf{q}}x^*(\mathbf{q}) = D_q F(\mathbf{x}; \mathbf{q})$$

- From FOC we get $D_{\mathbf{x}}\phi(\mathbf{x}; \mathbf{q}) = \boldsymbol{\lambda}^\top D_{\mathbf{x}}F(\mathbf{x}; \mathbf{q})$
- Thus,

$$\left. D_{\mathbf{x}}\phi(\mathbf{x}, \mathbf{q}) \right|_{\mathbf{q}=\overline{\mathbf{q}},\ \mathbf{x}=x^*(\overline{\mathbf{q}})} D_{\mathbf{q}}x^*(\mathbf{q}) = \boldsymbol{\lambda}^\top D_{\mathbf{x}}F(\mathbf{x}; \mathbf{q}) D_{\mathbf{q}}x^*(\mathbf{q}) = -\boldsymbol{\lambda}^\top D_q F(\mathbf{x}; \mathbf{q})$$

Envelope Theorem With Constraints

- The change in the objective function due to a change in $\mathbf{q}$ is:

$$D_{\mathbf{q}}\phi(x^*(\overline{\mathbf{q}}); \overline{\mathbf{q}}) = \left. D_{\mathbf{q}}\phi(\mathbf{x}, \mathbf{q}) \right|_{\mathbf{q}=\overline{\mathbf{q}}, \mathbf{x}=x^*(q)} - \left(\lambda^*(\overline{\mathbf{q}})\right)^\top \left. D_{\mathbf{q}}F(\mathbf{x}, \mathbf{q}) \right|_{\mathbf{q}=\overline{\mathbf{q}}, \mathbf{x}=x^*(\bar{q})}$$

Next Class

- Applications of Envelope Theorem
- Hicksian Demand
- Duality
- Slutsky Decomposition: Income and Substitution Effects